

# Cubic Graphs, Disjoint Matchings and Some Inequalities

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## Abstract

For  $k = 2, 3$  and a cubic graph  $G$  let  $\nu_k(G)$  denote the size of a maximum  $k$ -edge-colorable subgraph of  $G$ . Mkrtchyan, Petrosyan and Vardanyan proved that  $\nu_2(G) \geq \frac{4}{5} \cdot |V(G)|$ ,  $\nu_3(G) \geq \frac{7}{6} \cdot |V(G)|$  [13]. They were also able to show that  $\nu_2(G) + \nu_3(G) \geq 2 \cdot |V(G)|$  [3] and  $\nu_2(G) \leq \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}$  [13]. In the present work, we show that the last two inequalities imply the first two of them. Moreover, we show that  $\nu_2(G) \geq \alpha \cdot \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}$ , where

$\alpha = \frac{16}{17}$ , if  $G$  is a cubic graph,

$\alpha = \frac{20}{21}$ , if  $G$  is a cubic graph containing a perfect matching,

$\alpha = \frac{44}{45}$ , if  $G$  is a bridgeless cubic graph.

Finally, we investigate the parameters  $\nu_2(G)$  and  $\nu_3(G)$  in the class of claw-free cubic graphs. We improve the lower bounds for  $\nu_2(G)$  and  $\nu_3(G)$  for claw-free bridgeless cubic graphs to  $\nu_2(G) \geq \frac{29}{30} \cdot |V(G)|$ ,  $\nu_3(G) \geq \frac{43}{45} \cdot |E(G)|$ . We also show that  $\nu_2(G) \geq \frac{35}{36} \cdot |V(G)|$  when  $n \geq 48$ . On the basis of these inequalities we are able to improve the coefficient  $\alpha$  for bridgeless claw-free cubic graphs.

*Keywords:*

Cubic graph, Bridgeless cubic graph, Claw-free cubic graph, Claw-free bridgeless cubic graph, Pair and triple of matchings, Edge-coloring, Parsimonious edge-coloring

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## 1. Introduction

In this paper graphs are assumed to be finite, undirected and without loops, though they may contain multi-edges. We will also consider simple graphs, which contain neither loops nor multi-edges.

The set of vertices and edges of a graph  $G$  will be denoted by  $V(G)$  and  $E(G)$ , respectively. Sometimes we will denote  $|V(G)|$  by  $n$ .

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Throughout this paper, we will investigate cubic graphs. A graph is *cubic* if every vertex is incident to exactly three edges.

A *matching* in a graph is a set of edges without common vertices. A matching, which covers all vertices of the graph, is called a *perfect matching*. A *k-factor* of a graph is a spanning *k*-regular subgraph. In particular, the edge-set of a 1-factor is a perfect matching. Moreover, a 2-factor is a set of cycles in the graph that covers all its vertices. We will denote the smallest possible number of odd cycles in a 2-factor of a cubic graph  $G$  by  $\omega(G)$ .

A part of paper works with subclass of cubic graphs which are called *claw-free* cubic graphs. A graph is *claw-free* if it has no induced subgraph isomorphic to  $K_{1,3}$ .

A graph  $G$  is called *k-edge colorable*, if its edges can be assigned  $k$  colors so that adjacent edges receive different colors. A subgraph  $H$  of a graph  $G$  is called *maximum k-edge-colorable*, if  $H$  is *k*-edge-colorable and contains maximum number of edges. If  $H$  is a *k*-edge-colorable subgraph of  $G$  and  $e \notin E(H)$ , then we will say that  $e$  is an *uncolored* edge with respect to  $H$ . If it is clear from the context with respect to which subgraph an edge is uncolored, we will avoid mentioning the subgraph.

By a classical result due to Shannon [17, 20, 22], we have that cubic graphs are 4-edge-colorable. It is an interesting and useful problem to investigate the sizes of subgraphs of cubic graphs that are colorable only with 1, 2 or 3 colors.

For  $k = 1, 2, 3$  and a cubic graph  $G$  let

$$\nu_k(G) = \max\{|E(H)| : H \text{ is a } k\text{-edge-colorable subgraph of } G\}.$$

The *resistance*  $r_3(G)$  of  $G$  is the minimum of number of edges that have to be removed from  $G$  in order to obtain a 3-edge-colorable graph. Note that  $r_3(G) = |E(G)| - \nu_3(G)$ .

Albertson and Haas [1, 2], Steffen [18, 19] and Mkrtchyan et al. [13] investigated the lower bounds for  $\frac{\nu_k(G)}{|V(G)|}$  in cubic graphs. As a result, in [13] an interesting relation between  $\nu_2(G)$  and  $\nu_3(G)$  is proved, which states that for any cubic graph  $G$

$$\nu_2(G) \leq \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}.$$

The problem has been investigated in [4, 9, 14, 15, 23] when  $k = 1$ , and for regular graphs of high girth in [6].

The problem has also been investigated in the case when the graphs need not be cubic [7, 12, 16].

In the present work we give short proofs of main results of Mkrtchyan et al. [13]. We also prove lower bounds for  $\nu_2(G)$  in terms of  $|V(G)|$  and  $\frac{|V(G)| + 2 \cdot \nu_3(G)}{4}$  in the following sub-classes of cubic graphs:

1. (a) cubic graphs  
(b) cubic graphs containing a perfect matching  
(c) bridgeless cubic graphs
2. (a) claw-free cubic graphs  
(b) claw-free bridgeless cubic graphs

In some cases our lower bounds are best-possible. Terms and concepts that we do not define, can be found in [8, 24].

## 2. Inequalities and bounds for cubic graphs

First we formulate a proposition that will be helpful for our presentation of results. It has been applied already in [19] for bridgeless cubic graphs. Here we state and prove it for general graphs.

**Proposition 2.1.** *For any graph  $G$*

$$\nu_2(G) \geq \frac{2}{3} \cdot \nu_3(G).$$

*Proof.* Let  $(H, H', H'')$  be a triple of edge-disjoint matchings of  $G$  with  $|H| + |H'| + |H''| = \nu_3(G)$ . Obviously, the following inequalities are true:

$$\nu_2(G) \geq |H| + |H'|,$$

$$\nu_2(G) \geq |H'| + |H''|,$$

$$\nu_2(G) \geq |H| + |H''|.$$

Summing up these inequalities, we get:

$$3 \cdot \nu_2(G) \geq 2 \cdot \nu_3(G),$$

or

$$\nu_2(G) \geq \frac{2}{3} \cdot \nu_3(G).$$

The proof of the proposition is complete. □

In [13] Mkrtchyan, Petrosyan and Vardanyan proved that

**Theorem 2.1.** *For any cubic graph  $G$*

- (1)  $\nu_2(G) \geq \frac{4}{5} \cdot |V(G)|,$
- (2)  $\nu_3(G) \geq \frac{7}{6} \cdot |V(G)|,$
- (3)  $\nu_2(G) + \nu_3(G) \geq 2 \cdot |V(G)|,$
- (4)  $\nu_2(G) \leq \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}.$

The proofs of (1) and (2) of Theorem 2.1 given in [13] is long. Here we show that (3) and (4) imply (1) and (2).

**Theorem 2.2.** *For every cubic graph  $G$*

$$\nu_2(G) \geq \frac{4}{5} \cdot |V(G)|.$$

*Proof.* Due to (3) of Theorem 2.1, we have

$$\nu_2(G) + \nu_3(G) \geq 2 \cdot |V(G)|$$

and therefore

$$\frac{2}{3} \cdot \nu_2(G) + \frac{2}{3} \cdot \nu_3(G) \geq \frac{4}{3} \cdot |V(G)|.$$

We have also the following inequality (Proposition 2.1):

$$\nu_2(G) \geq \frac{2}{3} \cdot \nu_3(G).$$

So, it follows:

$$\frac{5}{3} \cdot \nu_2(G) \geq \frac{4}{3} \cdot |V(G)|,$$

or equivalently,

$$\nu_2(G) \geq \frac{4}{5} \cdot |V(G)|.$$

The proof of Theorem 2.2 is complete. □

**Theorem 2.3.** *For every cubic graph  $G$*

$$\nu_3(G) \geq \frac{7}{6} \cdot |V(G)|.$$

*Proof.* Due to (3) of Theorem 2.1, we have

$$\nu_2(G) + \nu_3(G) \geq 2 \cdot |V(G)|.$$

(4) of Theorem 2.1 states:

$$\nu_2(G) \leq \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}.$$

So, we have:

$$\frac{|V(G)| + 2 \cdot \nu_3(G)}{4} + \nu_3(G) \geq 2 \cdot |V(G)|,$$

or

$$|V(G)| + 2 \cdot \nu_3(G) + 4 \cdot \nu_3(G) \geq 8 \cdot |V(G)|,$$

hence,

$$\nu_3(G) \geq \frac{7}{6} \cdot |V(G)|.$$

The proof of Theorem 2.3 is complete. □

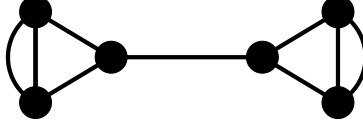


Figure 1: An example attaining the bound of Theorem 2.3.

The following graph on 6 vertices is a tight example for this inequality (Figure 1).

(4) of Theorem 2.1 provides an upper bound for  $\nu_2(G)$  in terms of  $\frac{|V(G)| + 2 \cdot \nu_3(G)}{4}$ . Here we address the problem of finding a lower bound for  $\nu_2(G)$  in terms of the same expression. We investigate this problem in the class of cubic graphs, the class of cubic graphs containing a perfect matching and the class of bridgeless cubic graphs.

Our first result states:

**Theorem 2.4.** *For any cubic graph  $G$*

$$\nu_2(G) \geq \frac{16}{17} \cdot \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}.$$

*Proof.* Due to Theorem 2.2, we have

$$\nu_2 \geq \frac{4}{5} \cdot n,$$

which is the same as

$$5 \cdot \nu_2 \geq 4 \cdot n.$$

Therefore we can write the following chain of inequalities:

$$17 \cdot \nu_2(G) \geq 4 \cdot n + 12 \cdot \nu_2(G) = 4 \cdot n + 8 \cdot \frac{3}{2} \nu_2(G) \geq 4 \cdot n + 8 \nu_3(G)$$

The last inequality  $\frac{3}{2} \cdot \nu_2(G) \geq \nu_3(G)$  follows from Proposition 2.1. Then,

$$17 \cdot \nu_2(G) \geq 4 \cdot (n + 2 \nu_3(G)).$$

We can write the final result in the following form:

$$\nu_2(G) \geq \frac{16}{17} \cdot \frac{n + 2 \cdot \nu_3(G)}{4}.$$

The proof of Theorem 2.4 is complete. □

The Sylvester graph on 10 vertices is a tight example for this inequality (Figure 2). For cubic graphs containing a perfect matching, we are able to improve the proved lower bound. The proof of this result requires the following auxiliary lemma.

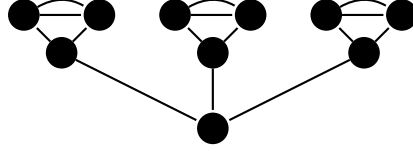


Figure 2: An example attaining the bound of Theorem 2.4.

**Lemma 2.1.** *For any cubic graph  $G$  containing a perfect matching*

$$\nu_2(G) \geq \frac{5}{6} \cdot n.$$

*Proof.* Let  $F$  be a perfect matching of  $G$ , and let  $\omega(\bar{F})$  be the number of odd cycles in the 2-factor  $G - F$ . A 2-edge-colorable subgraph of  $G$  can be obtained by taking  $F$  and a maximum matching in  $G - F$ . Hence, we have

$$\nu_2(G) \geq \frac{n}{2} + \frac{n - \omega(\bar{F})}{2}.$$

Since the length of each odd cycle of  $\bar{F}$  is at least 3, we have

$$\omega(\bar{F}) \leq \frac{n}{3}.$$

Hence,

$$\nu_2(G) \geq \frac{n}{2} + \frac{n}{3} = \frac{5}{6} \cdot n.$$

The proof of Lemma 2.1 is complete. □

We are ready to prove the main theorem for the class of cubic graphs containing a perfect matching.

**Theorem 2.5.** *For any cubic graph  $G$  containing a perfect matching*

$$\nu_2(G) \geq \frac{20}{21} \cdot \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}.$$

*Proof.* From Lemma 2.1 we have

$$6 \cdot \nu_2(G) \geq 5 \cdot n$$

$$21 \cdot \nu_2(G) \geq 5 \cdot n + 15 \cdot \nu_2(G) = 5 \cdot n + 10 \cdot \frac{3}{2} \cdot \nu_2(G) \geq 5 \cdot n + 10 \cdot \nu_3(G)$$

The last inequality  $\frac{3}{2} \cdot \nu_2(G) \geq \nu_3(G)$  follows from Proposition 2.1. Then,

$$21 \cdot \nu_2(G) \geq 5 \cdot (n + 2\nu_3(G)).$$

We can write the final result in the following form:

$$\nu_2(G) \geq \frac{20}{21} \cdot \frac{n + 2 \cdot \nu_3(G)}{4}.$$

The proof of Theorem 2.5 is complete. □

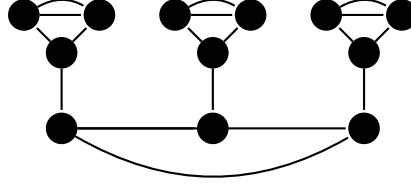


Figure 3: An example attaining the bound of Theorem 2.5.

The graph from Figure 3 attains the bound of Theorem 2.5.

Petersen theorem states that any bridgeless cubic graph contains a perfect matching [24]. Hence, one can claim that

$$\nu_2(G) \geq \frac{20}{21} \cdot \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}$$

for this class of graphs. It turns out that no bridgeless cubic graph can attain this bound. In other words, we are able to improve the coefficient  $\frac{20}{21}$  in this class.

Our proof will require the following proposition, which is easy to see to be true. It implicitly makes use of the fact, that there is no a bridgeless cubic graph  $G$  with  $r_3(G) = 1$  [18, 19].

**Proposition 2.2.** *Let  $G$  be a bridgeless cubic graph.*

(1) *If  $r_3(G) \leq 2$ , then*

$$\nu_2(G) = \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}.$$

(2) *If  $r_3(G)$  is odd, then*

$$\nu_2(G) < \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}.$$

Our main result states:

**Theorem 2.6.** *For any bridgeless cubic graph  $G$*

$$\nu_2(G) \geq \frac{44}{45} \cdot \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}.$$

*Proof.* If  $n \leq 10$ , then it is known that  $r_3(G) \leq 2$ . Hence, (1) of Proposition 2.2 implies that  $G$  satisfies the statement of the theorem. Thus without loss of generality, we can assume that  $n \geq 12$ .

Steffen in [19] proved that  $\nu_2(G) \geq \frac{11}{12} \cdot n$  when  $n \geq 12$ . Hence,

$$12 \cdot \nu_2(G) \geq 11 \cdot n$$

$$45 \cdot \nu_2(G) \geq 11 \cdot n + 33 \cdot \nu_2(G) = 11 \cdot n + 22 \cdot \frac{3}{2} \cdot \nu_2(G) \geq 11 \cdot n + 22 \cdot \nu_3(G).$$

The last inequality  $\frac{3}{2} \cdot \nu_2(G) \geq \nu_3(G)$  follows from Proposition 2.1. Then,

$$45 \cdot \nu_2(G) \geq 11 \cdot (n + 2\nu_3(G)).$$

We can write the final result in the following form:

$$\nu_2(G) \geq \frac{44}{45} \cdot \frac{n + 2 \cdot \nu_3(G)}{4}.$$

The proof of Theorem 2.6 is complete. □

We are not able to exhibit a bridgeless cubic graph attaining this bound. Moreover, we suspect that

**Conjecture 2.1.** *For any bridgeless cubic graph  $G$*

$$\nu_2(G) \geq \frac{52}{53} \cdot \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}.$$

Using the results of [5], we can show that this conjecture holds for any bridgeless cubic graph with  $|V(G)| \leq 26$ . In [5] it is shown that any connected non-3-edge-colorable bridgeless cubic graph  $G$  contains a vertex  $w$  such that  $G - w$  is Hamiltonian. One can easily see that this implies that  $r_3(G) \leq 2$ . Hence,  $\nu_2(G) = \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}$  due to (1) of Proposition 2.2.

The coefficient  $\frac{52}{53}$  is best-possible in the above conjecture, as the graph from Figure 4 attains it. The graph has 28 vertices and it is constructed as follows: we take 3 vertex disjoint copies of Petersen graph without a vertex (see left of Figure 4) and connect them according to the right of Figure 4.

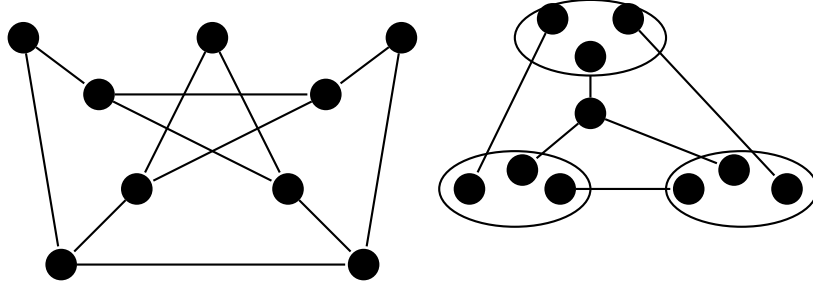


Figure 4: An example attaining the bound of Conjecture 2.1.

Note that all these coefficients  $\frac{44}{45}$ ,  $\frac{52}{53}$  are very close to 1, and there are also a vast number of graphs for which mentioned coefficient is 1, i.e.  $\nu_2(G) = \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}$ . So, it is an interesting problem to characterize the class of bridgeless cubic graphs  $G$  with  $\nu_2(G) = \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}$ .

We suspect that

**Conjecture 2.2.** *It is NP-hard to test whether a given bridgeless cubic graph  $G$  satisfies  $\nu_2(G) = \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}$ .*



### 3. Inequalities and bounds for claw-free cubic graphs

This section deals with lower bounds of  $\nu_2(G)$  and  $\nu_3(G)$  in the class of claw-free cubic graphs. We show that there exist substantial improvements for most of the inequalities proved in the previous section. On the other hand, we demonstrate that some of them cannot be improved.

Before we formulate the new inequalities let us give some definitions.

**Definition 3.1.** A subgraph of  $G$  is called a **diamond** if it is isomorphic to  $K_4 - e$ .

**Definition 3.2.** A **string of diamonds** is a maximal sequence  $D_1, D_2, \dots, D_k$  of diamonds in which, for every  $i \in \{1, 2, \dots, k-1\}$ ,  $D_i$  has a vertex adjacent to a vertex in  $D_{i+1}$ .

**Definition 3.3.** A connected claw-free cubic graph in which every vertex is in a diamond is called a **ring of diamonds**.

**Definition 3.4.** Replacing a vertex  $v$  with a triangle in a cubic graph is to replace  $v$  with three vertices  $v_1, v_2, v_3$  forming a triangle so that if  $e_1, e_2, e_3$  are three edges incident with  $v$ , then  $e_1, e_2, e_3$  will be incident with  $v_1, v_2, v_3$  respectively.

If a graph  $G$  is obtained from the graph  $H$  by replacing all vertices of  $H$  with a triangle, then we will write  $G = H_\Delta$ .

We are ready to state the characterization of simple claw-free bridgeless cubic graphs proved by Sang-il Oum in [10].

**Theorem 3.1** ([10]). A graph  $G$  is a simple 2-edge-connected claw-free cubic if and only if either

- (i)  $G$  is isomorphic to  $K_4$ , or
- (ii)  $G$  is a ring of diamonds, or
- (iii)  $G$  can be built from a 2-edge-connected cubic graph  $H$  by replacing some edges of  $H$  with strings of diamonds and replacing each vertex of  $H$  with a triangle.

Let us also recall the following classical result of Sumner:

**Proposition 3.1.** ([21]) If  $G$  is a connected claw-free graph of even order, then  $G$  has a perfect matching.

We are ready to improve the lower bound for  $\nu_2(G)$  in the class of claw-free cubic graphs.

**Theorem 3.2.** For any claw-free cubic graph  $G$

$$\nu_2(G) \geq \frac{5}{6} \cdot |V(G)|.$$

*Proof.* If  $G$  is not connected, then by proving the inequality for each connected component we will prove it for  $G$ . So, we can assume that  $G$  is connected. Proposition 3.1 implies that  $G$  has a perfect matching. Hence, by Lemma 2.1, the above inequality holds.  $\square$

Note that the lower bound for  $\nu_3(G)$  in claw-free cubic graphs coincides with the lower bound in general cubic graphs (Theorem 2.3). This follows from the observation that the tight example is a claw-free graph (see Figure 1).

Also, note that the inequality from Theorem 2.5 cannot be improved for claw-free cubic graphs, as the tight example is a claw-free graph as well.

Below we improve the lower bound for  $\nu_2(G)$  in the class of claw-free bridgeless cubic graphs. For that purpose we will require the following auxiliary results.

**Proposition 3.2.** ([13]) *Let  $a, b, c, d$  be positive real numbers and let  $\frac{a}{b} \geq \alpha$ ,  $\frac{c}{d} \geq \alpha$ . Then the following inequality holds:*

$$\frac{a+c}{b+d} \geq \alpha$$

**Lemma 3.1.** (Folklore, see also [11]) *If  $G$  is a bridgeless cubic graph, then  $G$  has a triangle-free 2-factor.*

**Lemma 3.2.** *Assume that  $G = H_\Delta$ , where  $H$  is a bridgeless cubic graph. Then*

$$\omega(G) \leq \frac{|V(G)|}{15}.$$

*Proof.* Due to Lemma 3.1,  $H$  has a triangle-free 2-factor  $F$ . Let us construct a 2-factor  $F'$  of  $G$ . Assume that  $F$  passes a vertex  $v$  of  $H$  and  $v$  is replaced with vertices  $v_1, v_2, v_3$  in  $G$ . Then  $F'$  will contain the edges  $e', v_1v_2, v_2v_3, f'$  (see Figure 5). By doing this for all vertices of  $F$ , we will get  $F'$ , which will be a 2-factor of  $G$ .

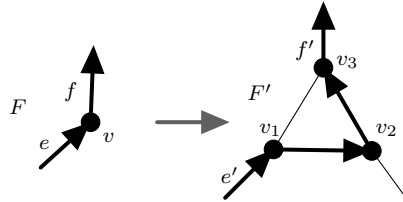


Figure 5: The triangle-free 2-factor  $F$  and  $F'$

As  $F$  is triangle-free, the length of any of its odd cycles is at least 5. Hence, the length of any odd cycle in  $F'$  will be at least  $5 + 2 \cdot 5 = 15$ . Since the cycles in  $F'$  are vertex disjoint, we have that  $F'$  contains at most  $\frac{|V(G)|}{15}$  odd cycles. Hence,  $\omega(G) \leq \frac{|V(G)|}{15}$ .  $\square$

**Theorem 3.3.** *For any claw-free bridgeless cubic graph  $G$*

$$\nu_2(G) \geq \frac{29}{30} \cdot |V(G)|.$$

*Proof.* The proof is by induction on the number of vertices of  $G$ . Obviously, when  $n = 2$ , our inequality is true. Assume that the inequality holds for all claw-free bridgeless cubic graphs containing less than  $n$  vertices, and let us consider the graph  $G$ . Let us show that without loss of generality we can assume that  $G$  contains no multi-edges. Suppose it has (Figure 6).

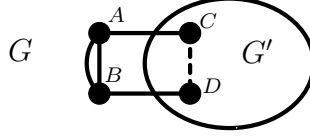


Figure 6: A multi-edge in  $G$ .

Construct a smaller graph  $G'$  from  $G$  by removing the vertices  $A, B$  and connecting the vertices  $C$  and  $D$  with an edge (see dashed line in Figure 6). Observe that  $G'$  is a claw-free bridgeless cubic graph. Hence, by induction the inequality holds for  $G'$ :

$$\frac{\nu_2(G')}{|V(G')|} \geq \frac{29}{30}.$$

It is not hard to see that

$$|V(G)| = |V(G')| + 2$$

and

$$\nu_2(G) \geq \nu_2(G') + 2.$$

Proposition 3.2 implies

$$\frac{\nu_2(G)}{|V(G)|} \geq \min \left\{ \frac{\nu_2(G')}{|V(G')|}, 1 \right\} \geq \frac{29}{30}.$$

Thus, without loss of generality, we can assume that  $G$  is a simple graph.

Due to Theorem 3.1, we will consider 3 cases.

1.  $G = K_4$ .

Obviously, the inequality holds for  $K_4$ :

$$\frac{\nu_2(G)}{|V(G)|} = 1 \geq \frac{29}{30}$$

2.  $G$  has a string of diamonds.

We observe that when  $G$  is a ring of diamonds,  $G$  fits this case.

Assume that  $G$  has a string of diamonds comprised of  $k$  diamonds (see Figure 7).

Consider a smaller graph  $G'$  obtained from  $G$  by removing the string of diamonds and adding the edge  $a$  like it is depicted in Figure 7.

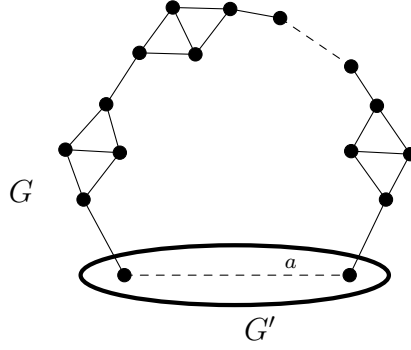


Figure 7: A string of diamonds in  $G$ .

Observe that

$$|V(G)| = |V(G')| + 4k.$$

Let us show that

$$\nu_2(G) \geq \nu_2(G') + 4k. \quad (1)$$

Let  $(H, H')$  be a pair of edge disjoint matchings of  $G'$ , such that their union forms a maximum 2-colorable subgraph of  $G'$ . We will consider 2 cases.

(a)  $a \notin H \cup H'$ .

In this case, we extend  $H$  and  $H'$  to a pair of edge disjoint matchings of  $G$  in the following way: add 2 distinct joint edges from every diamond to  $H$  and  $H'$  like it is shown in Figure 8.

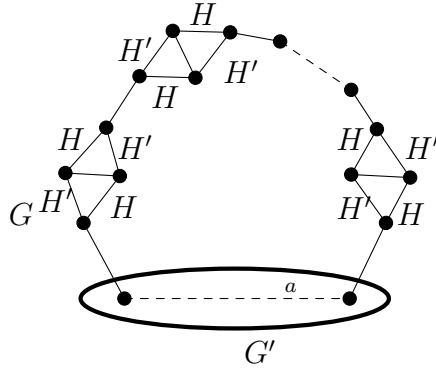


Figure 8: The case  $a \notin H \cup H'$ .

In this way we will add  $4k$  edges to  $H \cup H'$ . So, we can write:

$$\nu_2(G) \geq \nu_2(G') + 4k.$$

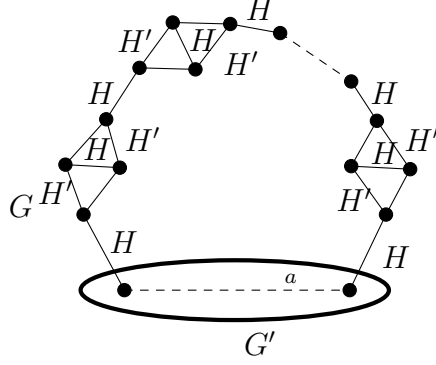


Figure 9: The case  $a \in H \cup H'$ .

(b)  $a \in H \cup H'$ .

Without loss of generality, we can assume that  $a \in H$ . In this case, we extend  $H$  and  $H'$  as follows: we remove  $a$  from  $H$  and add  $4k + 1$  edges in the way depicted in Figure 9.

Thus, in this case, we can write:

$$\nu_2(G) \geq (\nu_2(G') - 1) + 4k + 1 = \nu_2(G') + 4k.$$

The consideration of above 2 cases implies

$$\nu_2(G) \geq \nu_2(G') + 4k.$$

Due to Proposition 3.2 and induction hypothesis, we deduce

$$\frac{\nu_2(G)}{|V(G)|} \geq \min \left\{ \frac{\nu_2(G')}{|V(G')|}, 1 \right\} \geq \frac{29}{30}.$$

3.  $G = H_\Delta$ .

This means that  $G$  can be built from a 2-edge-connected cubic graph  $H$  by replacing each vertex of  $H$  with a triangle. We have

$$\nu_2(G) \geq \frac{|V(G)|}{2} + \frac{|V(G)| - \omega(G)}{2} = |V(G)| - \frac{\omega(G)}{2}.$$

Due to Lemma 3.2, we get:

$$\nu_2(G) \geq |V(G)| - \frac{|V(G)|}{30} = \frac{29}{30} \cdot |V(G)|.$$

The proof of the theorem is complete. □

**Theorem 3.4.** For any claw-free bridgeless cubic graph  $G$

$$\nu_3(G) \geq \frac{43}{45} \cdot |E(G)|.$$

*Proof.* Our proof is by induction on  $n$ . Obviously, our inequality is true when  $n = 2$ . By induction, assume that it is true for all claw-free bridgeless cubic graphs that have less than  $n$  vertices. Let us consider a claw-free bridgeless cubic graph  $G$  containing  $n \geq 4$  vertices.

Let us show that, without loss of generality, we can assume that  $G$  contains no multi-edge. Assume that it has (Figure 10).

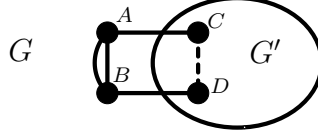


Figure 10: A multi-edge in  $G$ .

Consider a smaller graph  $G'$  obtained from  $G$  by removing vertices  $A, B$  and connecting vertices  $C$  and  $D$  with an edge (see dashed line in Figure 10).

By induction hypothesis, we have

$$\frac{\nu_3(G')}{|E(G')|} \geq \frac{43}{45}.$$

We also have

$$|E(G)| = |E(G')| + 3.$$

Let us show that

$$\nu_3(G) \geq \nu_3(G') + 3 \quad (2)$$

Suppose that  $(H, H', H'')$  is a triple of edge disjoint matchings of  $G'$ , such that their union forms a maximum 3-edge-colorable subgraph of  $G'$ . We will consider 2 cases.

(a)  $a \notin H \cup H' \cup H''$ .

It means that  $a$  is uncolored in  $G'$ . Hence, we can color  $G$  like it is shown in Figure 3.

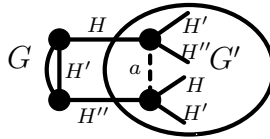


Figure 11:  $a \notin H \cup H' \cup H''$ .

In this way we have colored 3 more edges, which means that inequality (2) holds in this case.

(b)  $a \in H \cup H' \cup H''$ .

Without loss of generality, we can assume that  $a \in H$ . We can color  $G$  like it is depicted in Figure 12.

We have removed 1 edge and added 4 edges. Hence, inequality (2) holds in this case, too.

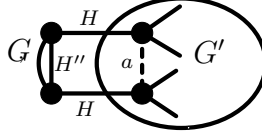


Figure 12:  $a \in H \cup H' \cup H''$ .

Due to Proposition 3.2 we deduce

$$\frac{\nu_3(G)}{|E(G)|} \geq \min \left\{ \frac{\nu_3(G')}{|E(G')|}, 1 \right\} \geq \frac{43}{45}.$$

Thus, without loss of generality, we can assume that  $G$  is a simple claw-free bridgeless cubic graph. Due to Theorem 3.1, we will consider 3 cases.

1.  $G = K_4$ .

Obviously, the inequality holds for  $K_4$ :

$$\frac{\nu_3(G)}{|E(G)|} = 1 \geq \frac{43}{45}.$$

2.  $G$  has a string of diamonds.

We observe that when  $G$  is a ring of diamonds,  $G$  fits this case. Suppose  $G$  has a string of diamonds comprised of  $k$  diamonds (see Figure 13).

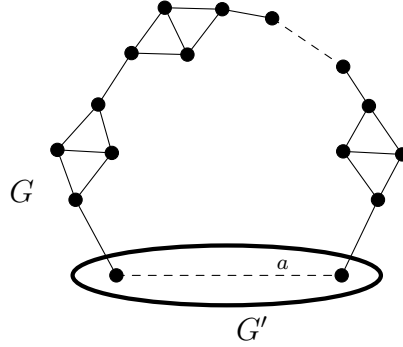


Figure 13: A string of diamonds in  $G$ .

Consider a smaller claw-free bridgeless cubic graph  $G'$  obtained from  $G$  by removing the string of diamonds and adding the edge  $a$  like it is depicted in Figure 13.

We have

$$|E(G)| = |E(G')| + 6k.$$

Let us show that

$$\nu_3(G) \geq \nu_3(G') + 6k. \tag{3}$$

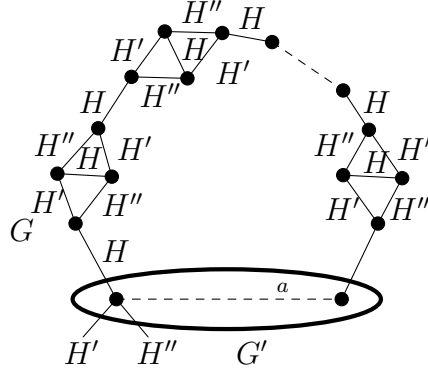


Figure 14:  $a \notin H \cup H' \cup H''$ .

Suppose that  $(H, H', H'')$  is a triple of edge disjoint matchings of  $G'$ , such that their union forms a maximum 3-edge-colorable subgraph of  $G'$ .

We will consider 2 cases.

- (a)  $a \notin H \cup H' \cup H''$ .

In this case, we can extend  $H, H'$  and  $H''$  to matchings of  $G$  in the way as it is shown in Figure 14.

So, in this case we can write:

$$\nu_3(G) \geq \nu_3(G') + 6k.$$

- (b)  $a \in H \cup H' \cup H''$ .

Assume that  $a \in H''$ . In this case, we remove the edge  $a$  from  $H''$  and extend  $H, H'$  and  $H''$  to matchings of  $G$  as it is shown in Figure 15. Observe that we have added  $6k + 1$  edges to  $H \cup H' \cup (H'' \setminus \{a\})$ .

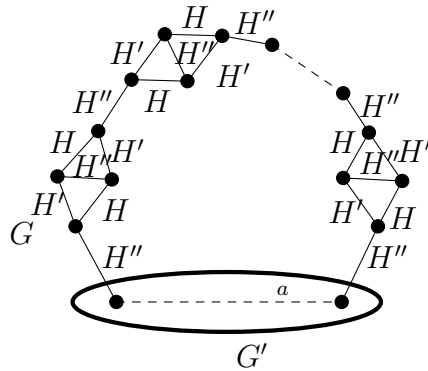


Figure 15:  $a \in H \cup H' \cup H''$ .

Hence, we can write:

$$\nu_3(G) \geq (\nu_3(G') - 1) + 6k + 1 = \nu_3(G') + 6k.$$



Due to Proposition 3.2 and induction hypothesis, we get:

$$\frac{\nu_3(G)}{|E(G)|} \geq \min \left\{ \frac{\nu_3(G')}{|E(G')|}, 1 \right\} \geq \frac{43}{45}.$$

3.  $G = H_\Delta$

This means that  $G$  can be built from a 2-edge-connected cubic graph  $H$  by replacing each vertex of  $H$  with a triangle. We have

$$\nu_3(G) \geq \frac{|V(G)|}{2} + 2 \cdot \frac{|V(G)| - \omega(G)}{2} = 3 \cdot \frac{|V(G)|}{2} - \omega(G).$$

Due to Lemma 3.2, we get:

$$\nu_3(G) \geq 3 \cdot \frac{|V(G)|}{2} - \frac{|V(G)|}{15} = \frac{43}{30} \cdot |V(G)| = \frac{43}{45} \cdot |E(G)|.$$

The proof of the theorem is complete.  $\square$

We observe that theorems 3.3 and 3.4 are best-possible in a sense that there is a graph attaining the bounds of these theorems. An example of such a graph is  $P_\Delta$ , where  $P$  is the Petersen graph.

For the proof of our next result, we will require some lemmas.

**Lemma 3.3.** *Let  $G'$  be a cubic graph, and assume that  $G$  is a cubic graph obtained from  $G'$  by replacing one of edges of  $G'$  with a string of diamonds. Then*

$$r_3(G) = r_3(G').$$

*Proof.* Assume that the string of diamonds of  $G$  that has replaced the edge  $a$  of  $G'$  contains exactly  $k$  diamonds. Then, as we have stated in the previous theorem, we have

$$|E(G)| = |E(G')| + 6k.$$

Taking into account that

$$r_3(G) = |E(G)| - \nu_3(G) \text{ and } r_3(G') = |E(G')| - \nu_3(G'),$$

it suffices to show that

$$\nu_3(G) = \nu_3(G') + 6k.$$

In strategy presented in the previous theorem can be used to prove that

$$\nu_3(G) \geq \nu_3(G') + 6k,$$

hence, we will only show that

$$\nu_3(G) \leq \nu_3(G') + 6k.$$

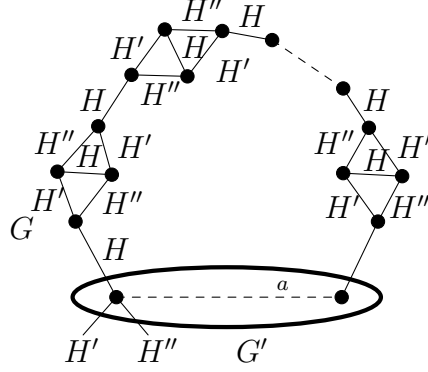


Figure 16: Restrictions of matchings form a 3-edge-colorable subgraph of  $G'$ .

Let  $(H, H', H'')$  be a triple of edge disjoint matchings of  $G$ , such that their union forms a maximum 3-edge-colorable subgraph of  $G$ . Observe that the string itself contains  $6k + 1$  edges of  $G$ . Now, if at least one of these edges of  $G$  does not belong to  $H \cup H' \cup H''$ , then the restrictions of these matchings to  $G'$  ( $H \cap E(G')$ ,  $H' \cap E(G')$ ,  $H'' \cap E(G')$ ) will form a 3-edge-colorable subgraph of  $G'$  (Figure 16), hence,

$$\nu_3(G') \geq |(H \cup H' \cup H'') \cap E(G')| \geq \nu_3(G) - 6k,$$

or

$$\nu_3(G) \leq \nu_3(G') + 6k.$$

Thus, without loss of generality, we can assume that all  $6k + 1$  edges of the string of  $G$  belong to  $H \cup H' \cup H''$ . Assume that the first edge of the string belongs to  $H''$  (Figure 17). Then, one can easily see that the string should be colored as on Figure 17. This coloring is unique up to flipping of edges of  $H$  and  $H'$  in the diamonds of the string.

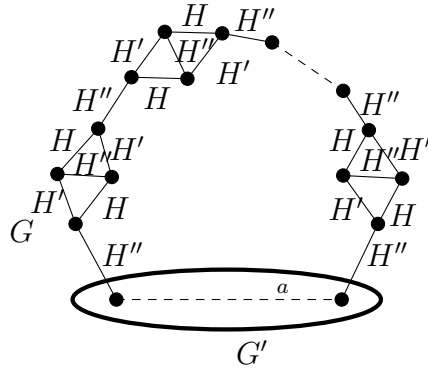


Figure 17: All edges of the string belong to the matchings.

Consider the restrictions of matchings of  $H$ ,  $H'$  and  $H''$  to  $G'$ , and add the edge  $a$  to  $H''$  (Figure 17). Observe that these new matchings will form a 3-edge-colorable subgraph of  $G'$ ,

hence,

$$\nu_3(G') \geq |(H \cup H' \cup H'') \cap E(G')| = (\nu_3(G) + 1) - 6k - 1 = \nu_3(G) - 6k,$$

or

$$\nu_3(G) \leq \nu_3(G') + 6k.$$

The proof of the lemma is complete.  $\square$

**Lemma 3.4.** (See the proof of Lemma 3.4 from [19]) Let  $G'$  be a bridgeless cubic graph, and assume that  $G$  is a bridgeless cubic graph obtained from  $G'$  by replacing one of vertices of  $G'$  with a triangle. Then

$$r_3(G) = r_3(G').$$

**Lemma 3.5.** [19] If  $G$  is a bridgeless cubic graph with at least 16 vertices, then

$$r_3(G) \leq \frac{|V(G)|}{8}.$$

**Lemma 3.6.** Let  $G$  be any claw-free bridgeless cubic graph with  $n \geq 48$ . Then

$$r_3(G) \leq \frac{|V(G)|}{24}.$$

*Proof.* If  $r_3(G) \leq 2$ , then

$$r_3(G) \leq 2 \leq \frac{|V(G)|}{24}.$$

Thus, without loss of generality, we can assume that  $r_3(G) \geq 3$ . If  $G$  contains multi-edges, then repeatedly remove the vertices of  $G$  adjacent to multi-edges and join the 2 degree-two vertices with an edge (Figure 10). We claim that the resulting graph  $G'$  contains no multi-edges and  $|V(G')| \geq 84$ .

If it contains a multi-edge, then one can easily see that  $r_3(G) = 0$  ( $G$  is 3-edge-colorable), which violates our assumption that  $r_3(G) \geq 3$ . Hence,  $G'$  is simple. Consider the Theorem 3.1. As  $r_3(G) \geq 3$ , we have that the theorem works from point (iii). Let  $H$  be the corresponding 2-edge-connected graph  $H$ . Lemmas 3.3 and 3.4 imply that  $3 \leq r_3(G') = r_3(H)$ .

Let us show that  $|V(H)| \geq 28$ . If  $|V(H)| \leq 26$ , then [5] implies that there is a vertex  $w$  of  $H$  such that  $H - w$  is Hamiltonian. One can easily see that this implies that  $r_3(H) \leq 2$  contradicting our assumption. Hence,  $|V(H)| \geq 28$ , which implies that  $|V(G')| \geq 3 \cdot 28 = 84$ .

Thus, without loss of generality, we can assume that our initial graph  $G$  is simple.

Similarly, one can show that  $G$  contains no string of diamonds. Thus, due to Theorem 3.1, there is a 2-edge-connected graph  $H$  such that  $G = H_\Delta$ . As  $|V(G)| \geq 48$ , we have  $|V(H)| \geq 16$ , hence, due to Lemma 3.5, we have

$$r_3(G) = r_3(H) \leq \frac{|V(H)|}{8} = \frac{|V(H_\Delta)|}{24} = \frac{|V(G)|}{24}.$$

The proof of the lemma is complete.  $\square$

**Theorem 3.5.** *For any claw-free bridgeless cubic graph  $G$  with  $n \geq 48$*

$$\nu_2(G) \geq \frac{35}{36} \cdot |V(G)|.$$

*Proof.* Due to Lemma 3.6

$$\nu_3(G) = |E(G)| - r_3(G) \geq \frac{3 \cdot |V(G)|}{2} - \frac{|V(G)|}{24} = \frac{35 \cdot |V(G)|}{24},$$

hence,

$$\nu_2(G) \geq \frac{2}{3} \cdot \nu_3(G) \geq \frac{2}{3} \cdot \frac{35|V(G)|}{24} = 35 \cdot \frac{|V(G)|}{36},$$

when  $|V(G)| \geq 48$ .

The proof of the theorem is complete. □

**Theorem 3.6.** *For any claw-free bridgeless cubic graph  $G$  with  $n \geq 48$*

$$\nu_2(G) \geq \frac{140}{141} \cdot \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}.$$

*Proof.* From the Theorem 3.5 we have

$$36 \cdot \nu_2(G) \geq 35 \cdot n,$$

hence,

$$141 \cdot \nu_2(G) \geq 35 \cdot n + 105 \cdot \nu_2(G) = 35 \cdot n + 70 \cdot \frac{3}{2} \cdot \nu_2(G) \geq 35 \cdot n + 70 \cdot \nu_3(G).$$

The last inequality follows from Proposition 2.1. Then,

$$141 \cdot \nu_2(G) \geq 35 \cdot (n + 2\nu_3(G)).$$

The final result we can write in the following form:

$$\nu_2(G) \geq \frac{140}{141} \cdot \frac{n + 2 \cdot \nu_3(G)}{4}.$$

The proof of the theorem is complete. □

We were unable to find a claw-free bridgeless cubic graph attaining the bound of the previous theorem. Moreover, we suspect that

**Conjecture 3.1.** *For any claw-free bridgeless cubic graph  $G$*

$$\nu_2(G) \geq \frac{164}{165} \cdot \frac{|V(G)| + 2 \cdot \nu_3(G)}{4}.$$

The bound presented by the previous conjecture is tight, in a sense, that there is a graph attaining it. That example is obtained from the graph from Figure 4 by replacing all its vertices with triangles.

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